An Amplitude Finite Element Formulation for Multiple-Scattering by a Collection of Convex Obstacles

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We present a multiple-scattering solver for non-convex geometries obtained as the union of a finite number of convex obstacles. The algorithm is a finite element reformulation of a high-frequency integral equation technique proposed previously. It is based on an iterative solution of the scattering problem, where each iteration leads to the resolution of a single scattering problem in terms of a slowly oscillatory amplitude.

***Index Terms*— Multiple electromagnetic scattering, short-wave problem, finite element methods, iterative techniques**

I. Introduction

Solving multiple-scattering problems at high frequencies is a challenging problem, especially when the wavelength is significantly smaller than the size of the scattering obstacles.

For non-convex geometries obtained as the union of a finite number of convex surfaces, an efficient algorithm was proposed in [1] based on three main elements: 1) an iteratively computable Neumann series for the currents induced on the scattering surfaces, which accounts rigorously for multiple scattering; 2) a generalized ansatz that allows for *a priori* determination of the highly oscillatory phase of the currents in each term of the series; and 3) use of the single-scattering boundary-integral solver from [2] for the efficient evaluation of each one of the terms in this series.

In this paper we present a reformulation of this algorithm using a finite element approach, which requires a fundamental rethinking of steps 2) and 3) since the fields are not only to be computed on the boundary of the scatterers but also in the volume. This new finite element approach exhibits many interesting features, amongst which possible extensions to non-homogeneous media and more complex geometries. Also, the proposed finite element formulation uses standard basis functions and can thus be easily implemented in existing finite element codes.

We start by reformulating the multiple scattering problem in terms of multiple single scattering problems in section II. We show in section III how this formulation can be solved iteratively. In section IV we apply the phase reduction technique to each single scattering problem, in order to reduce the computational cost of each iteration. Finally, section V contains two illustrative examples, for which several iterative solution techniques are compared.

II. Multiple Scattering as Coupled Single-Obstacle Scattering

We investigate the numerical solution of the time-harmonic acoustic scattering problem of a plane wave $u^{\text{inc}}(\boldsymbol{x}) = e^{ik\boldsymbol{\alpha}\cdot\boldsymbol{x}}$, $|\boldsymbol{\alpha}| = 1$, by a collection of impenetrable obstacles $\Omega_p^- \subset \mathbb{R}^2$, $p = 1,\dots,M$, with closed boundaries Γ_p, in dimension d. In two dimensions ($d = 2$) this is equivalent to solving either a TE- or TM-electromagnetic problem (with the field u standing for the z component of the electric or the magnetic field). The real wavenumber k is related to the wavelength λ by $\lambda = 2\pi/k$. Setting $\Omega^- = \cup_{p=1}^M \Omega_p^-$, $\Gamma = \cup_{p=1}^M \Gamma_p$ and $\Omega^+ = \mathbb{R}^2 \backslash \overline{\Omega^-}$, the boundary value problem reads:

$$\begin{aligned} \Delta u + k^2 u = 0 \quad &\text{in } \Omega^+, \\ u = -u^{\text{inc}} \text{ or } \partial_n u = -\partial_n u^{\text{inc}} \quad &\text{on } \Gamma, \\ \lim_{|\boldsymbol{x}|\to+\infty} |\boldsymbol{x}|^{(d-1)/2} (\nabla u \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - iku) = 0.& \end{aligned} \tag{1}$$

The multiple scattering problem (1) models the global scattering problem. It can be tackled using various numerical methods as-is. However, at both the theoretical and numerical levels, an interesting alternative is to reformulate the initial multiple scattering problem as M coupled single-obstacle scattering problems. Let us emphasize that this reduction is possible due to the linearity of the problem and holds for arbitrary shapes of the scatterers. We develop this point of view hereafter. This new formulation of the problem leads to a decomposition of the scattered field as

$$u = \sum_{p=1}^{M} u_p, \tag{2}$$

where each fictitious scattered wave u_p corresponds to the wave reflected by the scatterer p—and only by it—when it is illuminated simultaneously by the incident wave u^{inc} and the waves u_q, for $q = 1, ..., M$, with $q \neq p$.

The family of M coupled single-obstacle scattering problems for $p = 1, ..., M$, admits a unique solution $(u_1, \dots, u_M)$ satisfying [3]:

$$\begin{aligned} \Delta u_p + k^2 u_p = 0 \quad &\text{in } \mathbb{R}^d \setminus \overline{\Omega_p^-}, \\ u_p = -u^{\text{inc}} - \sum_{q=1,q\neq p}^{M} u_q \quad &\text{or} \\ \partial_{\mathbf{n}_{\Gamma_p}} u_p = -\partial_{\mathbf{n}_{\Gamma_p}} u^{\text{inc}} - \sum_{q=1,q\neq p}^{M} \partial_{\mathbf{n}_{\Gamma_p}} u_q \quad &\text{on } \Gamma_p, \\ \lim_{\|\mathbf{x}\|\to\infty} \|\mathbf{x}\|^{(d-1)/2} \left(\nabla u_p \cdot \frac{\mathbf{x}}{\|\mathbf{x}\|} - iku_p\right) = 0\,.& \end{aligned} \tag{3}$$

III. Iterative Solution of Multiple Scattering Problem

Instead of solving (3) directly, we look for the solution in terms of the series $u = \sum_{m=1}^{\infty} \sum_{p=1}^{M} u_p^{(m)}$, where $u_p^{(m)}$

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satisfies

$$\begin{aligned}
\Delta u_p^{(m)} + k^2 u_p^{(m)} = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{\Omega_p^-}, \\
u_p^{(m)} = s_p^{(m)} \text{ or } \partial_n u_p^{(m)} = \partial_n s_p^{(m)} \quad \text{on } \Gamma_p, \\
\lim_{|\boldsymbol{x}| \to +\infty} |\boldsymbol{x}|^{(d-1)/2} (\nabla u_p^{(m)} \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - iku_p^{(m)}) = 0,
\end{aligned} \tag{4}$$

with

$$s_p^{(m)} = \begin{cases} -u^{\text{inc}} - \sum_{q=1}^{p-1} u_q^{(m)} & \text{for } m = 1, \\ -\sum_{q=1}^{p-1} u_q^{(m)} - \sum_{q=p+1}^{M} u_q^{(m-1)} & \text{for } m > 1. \end{cases} \tag{5}$$

In other words, we perform a Gauss-Seidel-type iteration where at each step we solve a scattering problem around the single obstacle Ω_p^-, with the fields scattered from the other obstacles as boundary condition [4]. As each correction $u_p^{(m)}$ can be interpreted as the correction introduced by the m-th wave reflection [1], [4], the iteration can be stopped when the norm of all corrections at step m is smaller than a prescribed tolerance. A Jacobi-type iteration can be straightforwardly obtained by slightly modifying (5) as follows:

$$s_p^{(m)} = \begin{cases} -u^{\text{inc}} & \text{for } m = 1, \\ -\sum_{q=1, q \neq p}^{M} u_q^{(m-1)} & \text{for } m > 1. \end{cases} \tag{6}$$

More sophisticated iterative schemes can be used instead of Gauss-Seidel or Jacobi. To understand how, it is useful to reformulate (3) in operator form as

$$(I - A)U = -U^{\text{inc}}, \tag{7}$$

where the operator A has the following block structure

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} & \cdots & A_{1M} \\ A_{21} & 0 & A_{23} & \cdots & A_{2M} \\ A_{31} & A_{32} & 0 & \cdots & A_{3M} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & A_{M3} & \dots & 0 \end{bmatrix}, \tag{8}$$

with A_{ij} the operator that solves the single scattering problem $\Delta u + k^2 u = 0$ in $\mathbb{R}^d \setminus \overline{\Omega_j^-}$ with boundary condition $u = -U_j$ or $\partial_{\mathbf{n}_{\Gamma_j}} u = -\partial_{\mathbf{n}_{\Gamma_j}} U_j$ on Γ_j and returns the solution u on Γ_i. The linear system (7) can be solved iteratively e.g. with a Krylov subspace method like a preconditioned GMRES. An explicit expression of the iteration operator A (8) in the context of integral equations was given in [1].

IV. Amplitude Finite Element Formulation

A standard finite element code could be used to solve (4), but the cost of *each iteration* would be similar to the cost of solving the original problem (1) and this would thus present little practical interest. Actually, even solving a single problem can rapidly become prohibitively expensive as the mesh density depends on the frequency, with a least 10 points per wavelength (and even more at very high frequencies due to the so-called pollution effect [5]). However, if the obstacles Ω_p^- are *convex*, then each step in the iterative process can be accelerated with the phase reduction (PR) procedure proposed in [6], [5].

The idea of the PR-FEM is to approximate the phase of the solution $u_p^{(m)}$ to reformulate the problem in terms of a slowly oscillatory envelope $a_p^{(m)} = u_p^{(m)} e^{-ik\phi_p^{(m)}}$ in order to reduce pollution effects in the future FEM discretization. This approach thus involves two steps:

1) find an approximation $\phi_p^{(m)}$ of the phase of $u_p^{(m)}$ in the whole computational domain $\mathbb{R}^d \setminus \overline{\Omega_p^-}$;
2) use $\phi_p^{(m)}$ to solve the scattering problem in terms of a new slowly varying unknown $a_p^{(m)}$.

This has the advantage that the resulting formulation can be straightforwardly integrated into a classical finite element solver without any additional or new basis functions. Step 1) is solved through the solution of an evolution equation in the exterior domain. If Ω_p^- is convex then this evolution equation is fairly simple, and the process can be further decomposed into two steps: the proposition of an initial condition through the OSRC techniques [7] and the construction of a propagator using pseudo-differential operator techniques. Step 2) is direct since it is only a change of unknown into the standard variational formulation for $u_p^{(m)}$. This leads to a new variational equation of the following kind [6], [5]: find $a_p^{(m)} \in H^1(\mathbb{R}^d \setminus \overline{\Omega_p^-})$ such that

$$\begin{aligned}
&\int_\Omega \nabla a_p^{(m)} \cdot \nabla \bar{b} \, d\Omega + ik \int_\Omega a_p^{(m)} \nabla \phi_p^{(m)} \cdot \nabla \bar{b} \, d\Omega \\
&- ik \int_\Omega \nabla a_p^{(m)} \cdot \nabla \bar{\phi}_p^{(m)} \, \bar{b} \, d\Omega + k^2 \int_\Omega (|\nabla \phi_p^{(m)}|^2 - 1) \, a_p^{(m)} \, \bar{b} \, d\Omega \\
&+ \text{ABC}(a_p^{(m)}, \phi_p^{(m)}, b) = 0, \quad \forall b(\boldsymbol{x}) \in H_0^1(\mathbb{R}^d \setminus \overline{\Omega_p^-}),
\end{aligned} \tag{9}$$

with $\text{ABC}(a_p^{(m)}, \phi_p^{(m)}, \bar{b})$ an absorbing boundary condition, e.g. the Bayliss-Gunzburger-Turkel ABC [8], [5].

The finite element solution consists in introducing a discretization of the domain $\mathbb{R}^d \setminus \overline{\Omega_p^-}$ and solving for $a_p^{(m)}$ in a finite-dimensional space spanned by finite element basis functions. Given the slowly oscillatory nature of the unknown amplitude, it can be accurately represented with standard polynomial FE basis functions. Furthermore, a much coarser discretization can be used away from the boundary of the single obstacle Γ_p, e.g. 2 points per wavelength. On the boundary a refined grid must be used for the phase extraction [5], although independent surface and volume grids could be used, provided a projection step is introduced [9].

V. Numerical Test

As an example, we consider the scattering of a plane wave e^{ikx} by either two or four circular cylinders of unit radius R, separated by a distance $d = R$. We use standard first order basis functions, as well as a Bayliss-Gunzburger-Turkel-like radiation condition to truncate the infinite domain. In all considered cases a prescribed tolerance on the L^2-norm equal to 10^{-3} is used to stop the iterative process.

Fig. 1 shows, for $kR = 25$, the amplitude $a_p^{(m)}$ and the phase $\phi_p^{(m)}$ for $m = 1, 2, 6$ and $p = 1, 2$, as well as the final solution u with two and four cylinders. For example, Gauss-Seidel converges after 6 iterations with two cylinders and after 14 iterations with four cylinders. Of particular notice is that each term in the series is not highly oscillatory and can thus be computed on a coarser grid than the final solution. In this case we used about 2 points per wavelength to compute

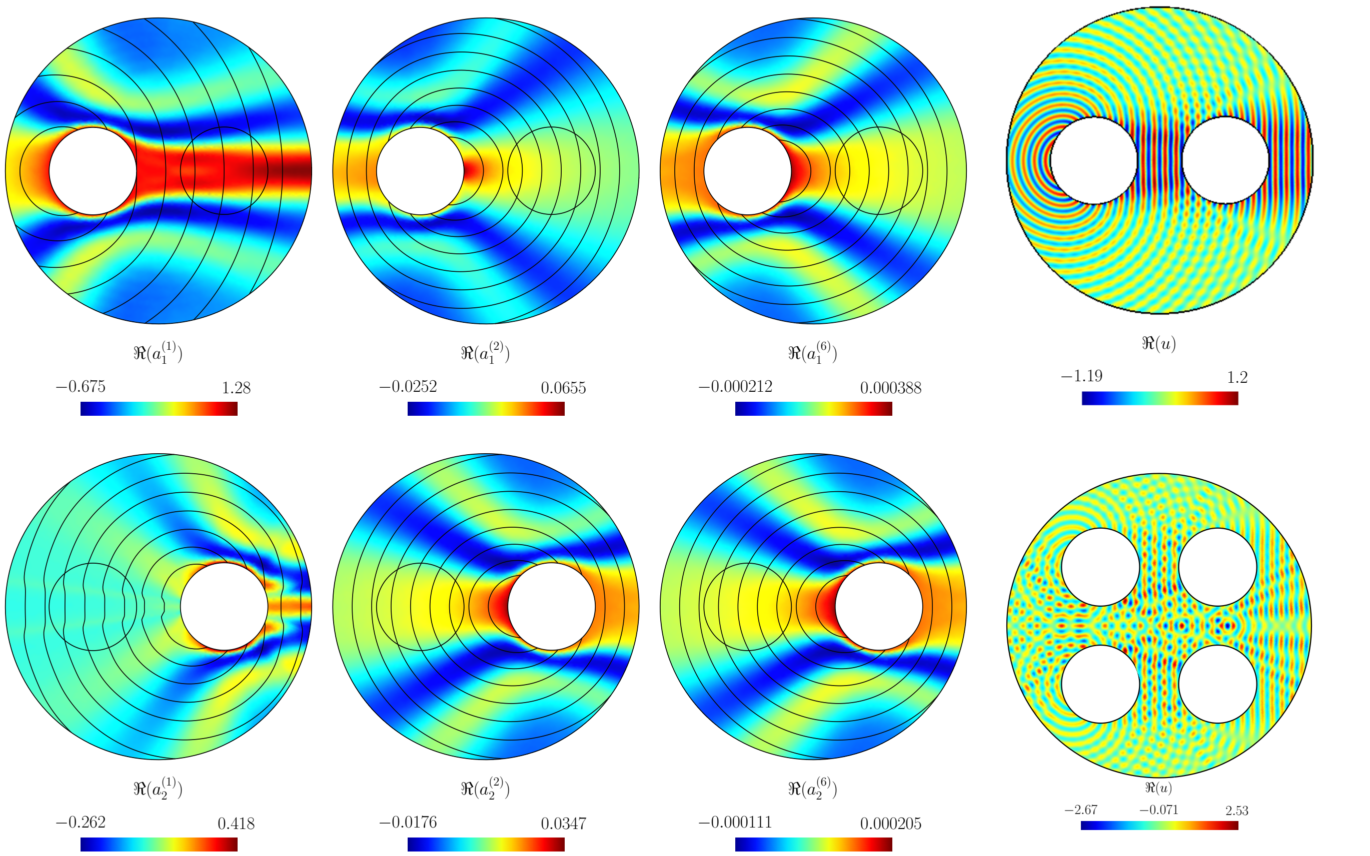

Fig. 1. Iterative solution around circular cylinders of unit radius R for an incident plane wave arriving from the left, with $kR = 25$. Left: Real part of the amplitudes $a_p^{(m)}$ for $p = 1$ and $p = 2$ (top to bottom) and $m = 1, 2, 6$ (left to right). On each graph iso-curves of the approximate phase $\phi_p^{(m)}$ are superimposed. Right (top): Real part of the final solution $u = \sum_{m=1}^{6} \sum_{p=1}^{2} a_p^{(m)} e^{i25\phi_p^{(m)}}$. Right (bottom): Real part of the final solution around four circular cylinders $u = \sum_{m=1}^{14} \sum_{p=1}^{4} a_p^{(m)} e^{i25\phi_p^{(m)}}$.

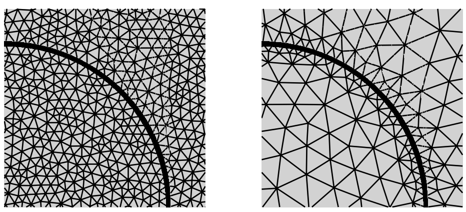

Fig. 2. Detail of mesh around one single obstacle required by the standard FE formulation (left) and the amplitude FE formulation (right) with $kR = 10$.

the approximate phase and the slowly oscillating amplitude, leading to a mesh containing 4300 nodes. The original problem would have required a mesh density of at least 10 points per wavelength, leading to about 25 times more unknowns (see Fig. 2).

Figs. 3 and 4 show the convergence of the iterative process with the standard FE and the new PR-FE formulation, respectively, for a two-cylinder single-row configuration and wavenumbers $k \in [0.1, 35]$. Results with GMRES, Gauss-Seidel and Jacobi iterative schemes are depicted. For the Jacobi and Gauss-Seidel iterative schemes, the convergence is slower at very low frequencies and diminishes to a constant number when k increases. As expected, the number of iterations required for achieving convergence when using a Jacobi scheme is roughly twice as much as when using a Gauss-Seidel scheme. With the standard FE formulation, applying GMRES reduces the number of iterations for the whole frequency range. With the new PR-FE formulation, the performance of the Krylov subspace method is excellent for low frequencies and it behaves at least as well as Gauss-Seidel scheme for higher frequencies.

The convergence for the four-cylinder configuration and k in $[0.1, 35]$ is depicted in Figs. 5 and 6 with the FE formulation and the PR-FE formulation, respectively. One can see that the iterative process diverges for low values of k with both the Gauss-Seidel and the Jacobi iterative schemes. In addition, with the standard FE formulation, the convergence of both Gauss-Seidel and Jacobi exhibits significant "bumps" for discrete frequencies, whereas GMRES converges smoothly. This is probably linked to the resonant frequencies of the structure. As shown in Fig. 6 the PR-FEM dramatically improves the convergence of all iterative methods at high frequencies.

VI. Conclusions

In this paper we have presented an iterative algorithm for computing the scattered field by a collection of convex obstacles using the finite element method. The formulation replaces the original multiple scattering problem by an iterative

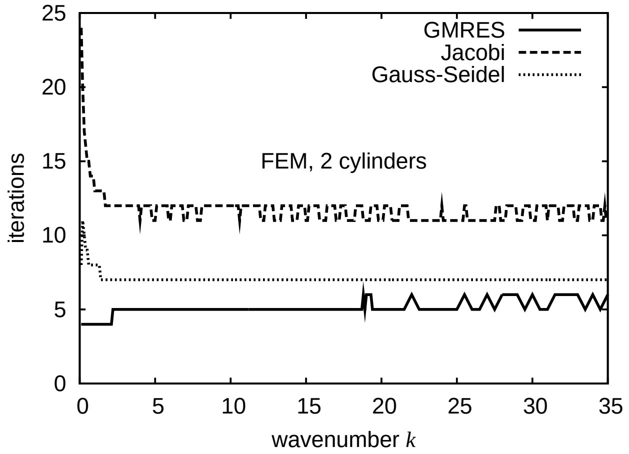

Fig. 3. Number of iterations for 2×1 circular cylinders versus wavenumber k. FEM with GMRES, Jacobi or Gauss-Seidel iterative scheme.

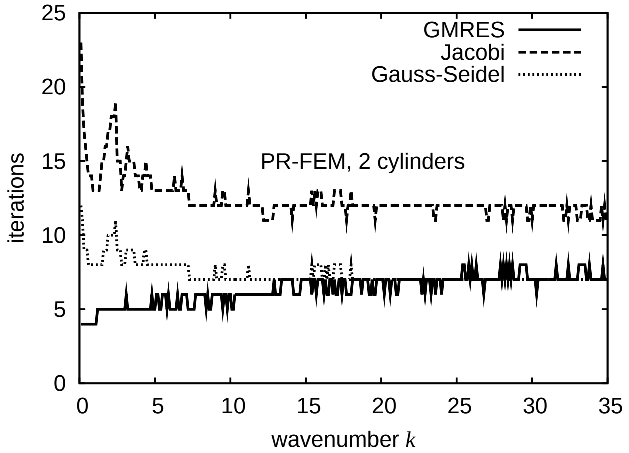

Fig. 4. Number of iterations for 2×1 circular cylinders versus wavenumber k. PR-FEM with GMRES, Jacobi or Gauss-Seidel iterative scheme.

series of single scattering problems. Each term in the series can be efficiently computed in terms of a slowly oscillatory amplitude that can be accurately represented on a coarse mesh. Three iterative schemes were compared and tested on both the original and the amplitude-based finite element formulations. GMRES proved to be the choice overall, leading to a number of iterations almost independent of the wavenumber.

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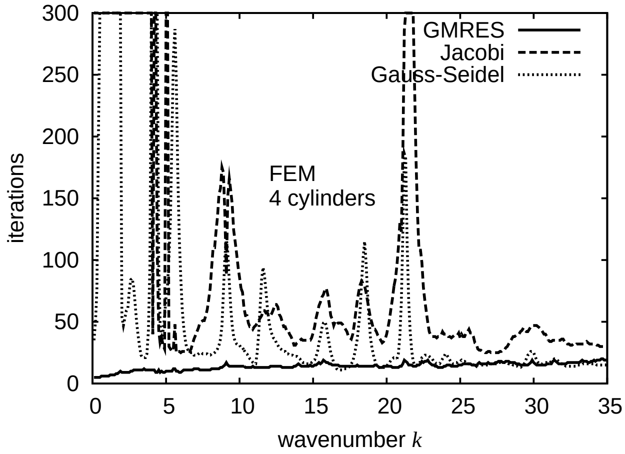

Fig. 5. Number of iterations for 2×2 circular cylinders versus wavenumber k. FEM with GMRES, Jacobi or Gauss-Seidel iterative scheme.

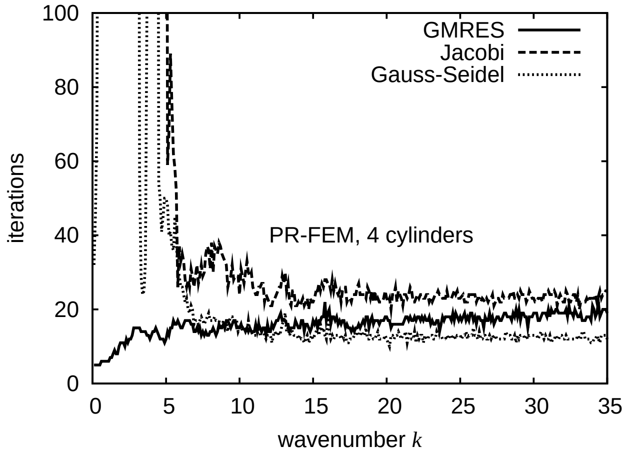

Fig. 6. Number of iterations for 2×2 circular cylinders versus wavenumber k. PR-FEM with GMRES, Jacobi or Gauss-Seidel iterative scheme.

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